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Extremal Kähler metrics and Hamiltonian functions

Thierry Chave [a], Henrik Pedersen [b,*], Christina Tønnesen-Friedman [c], Galliano Valent [a]

[a] *Laboratoire de Physique Théorique et des Hautes Energies, Unité Associée au CNRS URA 280, Université Paris 7, 2 Place Jussieu, 75251 Paris Cedex 05, France*

[b] *Department of Mathematics and Computer Science, Odense University, Campusvej 55, DK-5230 Odense M, Denmark*

[c] *Department of Mathematical Sciences, University of Aarhus, DK-8000 Århus C, Denmark*

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Abstract

Assuming a real torus acting on a Kähler manifold through holomorphic isometries, we obtain an ansatz for extremal Kähler metrics and we get new extremal metrics on some $\mathbb{CP}^1$-bundles over products of Kähler–Einstein manifolds of negative scalar curvature. We prove that one of the extremal metrics in four dimensions is locally conformally Einstein. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we construct new extremal Kähler metrics on certain $\mathbb{CP}^1$-bundles over products of Kähler–Einstein manifolds of negative scalar curvature. These metrics generalize recent extremal examples obtained on ruled surfaces [17].

We begin by constructing an ansatz for extremal metrics with commuting holomorphic isometries and we obtain a system of differential equations for the scalar curvature with respect to the Kähler quotient coordinates and the Hamiltonian functions. Such a Hamiltonian approach has been used earlier in studies of other Kähler geometries [11–15].

* Corresponding author. Present address: Department of Mathematics, University of Copenhagen, Universitetparken 5, 2100 Copenhagen, Denmark.

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Then we solve these equations in a special case which allows for a compactification to the projectivization $\mathbb{P}(\mathcal{O} \oplus \bigotimes_{i=1}^{n} K_i^{-1})$. Here the tensor product $\bigotimes_{i=1}^{n} K_i^{-1}$ is a pull back to the product $B_1 \times \cdots \times B_n$ where K_i^{-1} are the anti-canonical bundles of negative scalar curvature Kähler–Einstein manifolds B_i. The case where each B_i has positive scalar curvature is contained in the Koiso–Sakane approach [10] followed by Hwang [8].

Finally, in four dimensions, one of the metrics is locally conformally Einstein and we relate this fact to strongly extremal Kähler metrics [9,16].

2. An ansatz for extremal Kähler metrics

The notion of extremal Kähler metrics was introduced by Calabi [2–4]. On a compact manifold M^{2m} consider the functional $\mathcal{S}(\Omega) = \int_M s^2 \Omega^m$ where Ω is a Kähler form in a fixed Kähler class $[\Omega] \in H^2(M, \mathbb{R})$ and s is the scalar curvature of Ω. Critical points of $\mathcal{S}$ are called *extremal* Kähler metrics.

Extend the Riemannian metric g^{-1} on T^*M to a complex bilinear form on $T^*M \otimes \mathbb{C}$ and let $\sharp$ be the isomorphism $T^*M \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ given by $\alpha^\sharp = g^{-1}(\alpha, \cdot)$. Then g is extremal if and only if the (1,0)-vector field $(\bar{\partial} s)^\sharp$ is holomorphic. Alternatively, the gradient, grad s, of s is an infinitesimal automorphism of the complex structure J, i.e. the Lie-derivative $\mathcal{L}_{\operatorname{grad} s} J$ vanishes.

Following [15] we consider the situation with a real torus T^N acting freely on the Kähler manifold M^{2m} through holomorphic isometries.

Proposition 2.1. *Let (w_{ij}), $i, j = 1, \ldots, N$ and $(q_{\mu\nu})$, $\mu, \nu = 1, \ldots, m-N$ be positive definite matrices of smooth functions on an open set U in $\mathbb{C}^{m-N} \times \mathbb{R}^N$ with coordinates $(\xi^\mu = x^\mu + \mathrm{i} y^\mu, z^i)$. Let M be a T^N-bundle over U with connection 1-form $\omega = (\omega_1, \ldots, \omega_N) = (\mathrm{d}t_1 + \theta_1, \ldots, \mathrm{d}t_N + \theta_N)$ where $(t_1, \ldots, t_N)$ are coordinates on T^N and $\theta_i = A_{i\mu}\, \mathrm{d}x^\mu + B_{i\mu}\, \mathrm{d}y^\mu + C_{ij}\, \mathrm{d}z^j$ is defined on U. Suppose that*

$$\frac{\partial^2 q_{\mu\nu}}{\partial z^k \partial z^l} + \frac{\partial^2 w_{kl}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 w_{kl}}{\partial y^\mu \partial y^\nu} = 0, \tag{1}$$

$$\frac{\partial w_{kl}}{\partial z^j} = \frac{\partial w_{kj}}{\partial z^l}, \tag{2}$$

$$\frac{\partial C_{kl}}{\partial z^j} = \frac{\partial C_{kj}}{\partial z^l} \tag{3}$$

and assume the torus bundle has curvature

$$F_k = \frac{\partial q_{\mu\nu}}{\partial z^k} \mathrm{d}x^\mu \wedge \mathrm{d}y^\nu + \frac{\partial w_{kl}}{\partial x^\mu} \mathrm{d}y^\mu \wedge \mathrm{d}z^l + \frac{\partial w_{kl}}{\partial y^\mu} \mathrm{d}z^l \wedge \mathrm{d}x^\mu. \tag{4}$$

Then

$$g = q_{\mu\nu}(\mathrm{d}x^\mu\, \mathrm{d}x^\nu + \mathrm{d}y^\mu\, \mathrm{d}y^\nu) + w_{ij}\, \mathrm{d}z^i\, \mathrm{d}z^j + w^{ij} \omega_i \omega_j$$

is a Kähler metric on M. Conversely, any Kähler metric with a torus acting freely through holomorphic isometries can locally be constructed as above.

Proof. The proof is straightforward and we just explain a few things about the second part of the proposition. Suppose M is a T^N-symmetric Kähler manifold with metric g, Kähler form Ω and complex structure J and suppose $(X_1, \dots, X_N)$ are Hamiltonian vector fields generated by the torus action. Let $\mathrm{d}z^j = -\mathrm{i}_{X_j}\Omega$ define the Hamiltonian functions z^j. Then the metric is given as

$$g = q + w_{ij}\,\mathrm{d}z^i\,\mathrm{d}z^j + w^{ij}\omega_i\omega_j$$

where q is a Kähler metric on the quotient space of each level set of the Hamiltonians. Note that $w^{ij} = g(X_i, X_j)$. We have $g(JX_i, X_j) = -\,\mathrm{d}z^i(X_j)$ so $J\omega_i = -w_{ij}\,\mathrm{d}z^j$ and $\Omega = \Omega_q + \mathrm{d}z^k \wedge \omega_k$ where Ω_q is the Kähler form of the Kähler quotient. As J is integrable the exterior derivative $\mathrm{d}\varphi_k$ of the $(1,0)$-forms $\varphi_k = w_{kl}\,\mathrm{d}z^l + \mathrm{i}\omega_k$ must have no $(0,2)$ part. Also, for g to be Kähler we need $\mathrm{d}\Omega = 0$. These conditions are summed up in (2), and (3) and in the equation $\mathrm{d}\omega_k = F_k$ with F_k as in (4). Now, (1) is just the integrability condition $\mathrm{d}F_k = 0$. □

Remark 2.2. *Eq.* (4) *implies a series of identities. For example, there are three monopole-like equations*

$$\frac{\partial w_{kl}}{\partial x^\lambda} = \frac{\partial C_{kl}}{\partial y^\lambda} - \frac{\partial B_{k\lambda}}{\partial z^l}, \tag{5}$$

$$\frac{\partial w_{kl}}{\partial y^\lambda} = \frac{\partial A_{k\lambda}}{\partial z^l} - \frac{\partial C_{kl}}{\partial x^\lambda}, \tag{6}$$

$$\frac{\partial q_{\mu\nu}}{\partial z^k} = \frac{\partial B_{k\nu}}{\partial x^\mu} - \frac{\partial A_{k\mu}}{\partial y^\nu}, \tag{7}$$

and equations

$$\frac{\partial A_{l\mu}}{\partial x^\nu} = \frac{\partial A_{l\nu}}{\partial x^\mu}, \tag{8}$$

$$\frac{\partial B_{l\mu}}{\partial y^\nu} = \frac{\partial B_{l\nu}}{\partial y^\mu}. \tag{9}$$

Now, let M^{2m} be a T^N-symmetric Kähler metric as above. We look for the condition on the scalar curvature s for the metric to be extremal. We have the $(1,0)$-forms $\mathrm{d}\xi^u = \mathrm{d}x^u + \mathrm{i}\,\mathrm{d}y^u$ and $\mathrm{d}z^k - \mathrm{i}w^{kl}\omega_l$, so $\bar\partial s = \frac{1}{2}(\mathrm{d}s - \mathrm{i}J\,\mathrm{d}s)$ is given by

$$\bar\partial s = \frac{\partial s}{\partial \bar\xi^\mu}\,\mathrm{d}\bar\xi^\mu + \frac{1}{2}\frac{\partial s}{\partial z^k}(\mathrm{d}z^k - \mathrm{i}w^{kl}\omega_l). \tag{10}$$

Therefore we get

$$(\bar\partial s)^\sharp = \frac{\partial s}{\partial \bar\xi^\nu} q^{\mu\nu}\left(\frac{\partial}{\partial x^\nu} - \mathrm{i}J\frac{\partial}{\partial x^\nu}\right) - \left(A_{l\mu}\frac{\partial s}{\partial \bar\xi^\nu} q^{u\nu} + \frac{\mathrm{i}}{2}\frac{\partial s}{\partial z^l}\right)\left(\frac{\partial}{\partial t_l} - \mathrm{i}J\frac{\partial}{\partial t_l}\right).$$

We need to spell out the conditions for the vector field $(\bar\partial s)^\sharp$ to be holomorphic.

Lemma 2.3. *There exist smooth functions $F_{k\mu}$ such that the forms $\Phi_k = F_{k\mu}\,\mathrm{d}\xi^\mu + w_{kl}\,\mathrm{d}z^l + \mathrm{i}\omega_k$ together with $\mathrm{d}\xi^\mu$, $\mu = 1, \dots, m$ and $k = 1, \dots, N$, are a basis of holomorphic $(1,0)$-forms.*

Proof. Certainly $\mathrm{d}\xi^\mu$ is holomorphic and we look for holomorphic $(1,0)$-forms given as $\Phi_k = F_{k\mu}\,\mathrm{d}\xi^\mu + G^l_k\varphi_l$ for smooth functions $F_{k\mu}$, G^l_k and $\varphi_k = w_{kl}\,\mathrm{d}z^l + \mathrm{i}\omega_k$. Note that $\Phi_k(\partial/\partial t_l - \mathrm{i}J(\partial/\partial t_l))$ is holomorphic if Φ_k is holomorphic because $\partial/(\partial t_l)$ is a real holomorphic symmetry. However, $\Phi_k(\partial/(\partial t_l) - \mathrm{i}J(\partial/(\partial t_l))) = 2\mathrm{i}G^l_k$ so we already know that $\bar\partial G^l_k = 0$. We need $(\mathrm{d}\Phi_k)^{(1,1)} = 0$ and we find $(\mathrm{d}\Phi_k)^{(1,1)} = \bar\partial F_{k\mu} \wedge \mathrm{d}\xi^\mu + G^l_k\,\mathrm{d}\varphi_l^{(1,1)}$. Due to (2) and (4) we find

$$\mathrm{d}\varphi_l^{(1,1)} = -\frac{1}{2}\frac{\partial q_{\mu\nu}}{\partial z^l}\,\mathrm{d}\xi^\mu \wedge \mathrm{d}\bar\xi^\nu + \frac{\partial w_{lj}}{\partial \xi^\mu}\,\mathrm{d}\xi^\mu \wedge (\mathrm{d}z^j - \mathrm{i}w^{jp}\omega_p).$$

Put $G^l_k = \delta^l_k$. Then $(\mathrm{d}\Phi_k)^{(1,1)} = 0$ iff

$$\frac{\partial F_{k\mu}}{\partial z^j} - 2\frac{\partial w_{kj}}{\partial \xi^\mu} = 0, \tag{11}$$

$$\frac{\partial q_{\mu\nu}}{\partial z^k} + 2\frac{\partial F_{k\mu}}{\partial \bar\xi^\nu} = 0. \tag{12}$$

Now, the integrability condition for systems (11) and (12) is easily seen to be satisfied due to (1), (5) and (9). □

We are now ready to prove our ansatz. We refer to Proposition 2.1 for the notation.

Theorem 2.4. *Let M^{2m} be a T^N-symmetric compact Kähler manifold of scalar curvature s. Then the metric is extremal iff*

$$\frac{\partial}{\partial z^k}\left(q^{\mu\nu}\frac{\partial s}{\partial x^\mu}\right) = 0, \tag{13}$$

$$\frac{\partial}{\partial z^k}\left(q^{\mu\nu}\frac{\partial s}{\partial y^\mu}\right) = 0, \tag{14}$$

$$\frac{\partial}{\partial x^\lambda}\left(q^{\mu\nu}\frac{\partial s}{\partial x^\mu}\right) = \frac{\partial}{\partial y^\lambda}\left(q^{\mu\nu}\frac{\partial s}{\partial y^\mu}\right), \tag{15}$$

$$\frac{\partial}{\partial y^\lambda}\left(q^{\mu\nu}\frac{\partial s}{\partial x^\mu}\right) = -\frac{\partial}{\partial x^\lambda}\left(q^{\mu\nu}\frac{\partial s}{\partial y^\mu}\right), \tag{16}$$

$$\frac{\partial^2 s}{\partial z^k \partial z^l} + q^{\mu\nu}\frac{\partial w_{kl}}{\partial x^\nu}\frac{\partial s}{\partial x^\mu} + q^{\mu\nu}\frac{\partial w_{kl}}{\partial y^\nu}\frac{\partial s}{\partial y^\mu} = 0, \tag{17}$$

$$q^{\mu\nu}\frac{\partial s}{\partial x^\mu}\frac{\partial w_{kl}}{\partial y^\nu} = q^{\mu\nu}\frac{\partial s}{\partial y^\mu}\frac{\partial w_{kl}}{\partial x^\nu}. \tag{18}$$

Proof. $(\bar\partial s)^\sharp$ is holomorphic iff $\mathrm{d}\xi^\mu((\bar\partial s)^\sharp)$ and $\Phi_k((\bar\partial s)^\sharp)$ are holomorphic functions for all μ and k. We have seen in (10) that

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{\xi}^{\mu}} \mathrm{d}\bar{\xi}^{\mu} + \frac{1}{2} \frac{\partial f}{\partial z^k} (\mathrm{d}z^k - \mathrm{i} w^{kl} \omega_l)$$

for a function f. This leads to the six equations above. □

Obviously we need an expression for the scalar curvature in order to be able to work with the ansatz in Theorem 2.4. We have

Proposition 2.5. *Let M^{2m} be a symmetric Kähler metric as in Proposition* 2.1 *and let $u = \log\det q - \log\det w$. Then the scalar curvature s satisfies*

$$-s = \left\{ \frac{\partial^2 u}{\partial x^{\mu} \partial x^{\nu}} + \frac{\partial^2 u}{\partial y^{\mu} \partial y^{\nu}} + \left(w^{kl} \frac{\partial u}{\partial z^k} \right) \frac{\partial q_{\mu\nu}}{\partial z^l} \right\} q^{\mu\nu} + \frac{\partial}{\partial z^l} \left(w^{kl} \frac{\partial u}{\partial z^k} \right).$$

Proof. We have the Ricci form $\rho = -\frac{1}{2} \mathrm{d}J \,\mathrm{d} \log\det g$ and $\det g$ is the ratio between the volume and the $2m$ form associated with a holomorphic frame. From the identity $\Omega = \Omega_q + \mathrm{d}z^k \wedge \omega_k$ we get $\mathrm{vol} = (1/m!)\Omega^m = \mathrm{vol}(q) \wedge (\bigwedge_{k-1}^{N} \mathrm{d}z^k \wedge \omega_k)$. Furthermore, the $2m$-form Ψ associated to the holomorphic frame $\mathrm{d}\xi^{\mu}$, Φ_k satisfies $(i/2)^m \Psi = \det w \bigwedge_{k=1}^{N} \mathrm{d}z^k \wedge \omega_k \bigwedge_{\mu=1}^{m} \mathrm{d}x^{\mu} \wedge \mathrm{d}y^{\mu}$. Since $\mathrm{vol}(q) = \det q \wedge_{\mu=1}^{m} \mathrm{d}x^{\mu} \wedge \mathrm{d}y^{\mu}$ we get $\det g = \det q (\det w)^{-1}$. Furthermore, $2m\rho \wedge \Omega^{m-1} = s\Omega^m$ which after some elementary computations gives the claim. □

3. Examples of new extremal Kähler metrics

Assume $N = 1$, that is assume that we have an S^1-symmetric Kähler manifold M^{2m}. Assume, furthermore, that the Kähler quotient B^{2m-2} is a product $B_1^{2m_1} \times \cdots \times B_n^{2m_n}$ of compact Kähler–Einstein spaces (B_i, Q_i) of scalar curvature $-2m_i$ and that the Kähler metric q on B is equal to zQ where $Q = \sum_{i=1}^{n} \pi_i^* Q_i$ and $\pi_i \colon B \to B_i$ is the projection onto the ith factor. Then we have

$$\frac{\partial^2 \log\det Q_i}{\partial \xi_i^{\mu} \partial \bar{\xi}_i^{\nu}} = \frac{1}{2} (Q_i)_{\mu\nu},$$

where ξ_i^{μ} are complex coordinates on B_i. Also,

$$u = (m-1)\log z + \sum_{i=1}^{n} \log\det Q_i - \log w,$$

where u is defined in Proposition 2.5. It follows that

$$u_z = (m-1) z^{-1} - w^{-1} \frac{\partial w}{\partial z}$$

and that the scalar curvature is

$$s = -(m-1) \left(\frac{2}{z} + w^{-1} z^{-1} \frac{\partial u}{\partial z} \right) - \frac{\partial}{\partial z} \left(w^{-1} \frac{\partial u}{\partial z} \right).$$

The only equation on s left from Theorem 2.4 is $\partial^2 s/\partial z^2 = 0$, i.e. $s = Az + B$ for constants A, B of integration. Therefore $\varphi = w^{-1}\partial u/\partial z$ satisfies

$$z\frac{\partial\varphi}{\partial z} + \varphi(m-1) = -2(m-1) - Az^2 - Bz,$$
$$w^{-1}\frac{\partial w}{\partial z} = (m-1)z^{-1} - w\varphi.$$

We integrate these equations to get $w^{-1} = P(z)/z^{m-1}$ where

$$P(z) = \frac{-2}{m}z^m - c_1 z^{m+2} - c_2 z^{m+1} - c_3 z - c_4 \tag{19}$$

for c_i, $i = 1, 2, 3, 4$ constants of integration (compare this with the corresponding polynomials in [3,6]). Furthermore, from Eq. (4) we have $\mathrm{d}\omega = \sum_{i=1}^n \Omega_i$ where Ω_i is the Kähler form of B_i. The Chern form $c_1(K_i^{-1})$ of the anti-canonical bundle K_i^{-1} of B_i equals $\rho_i/2\pi$, where $\rho_i = -\Omega_i$ is the Ricci form of B_i. Thus, the manifold M of the Kähler metric $g = zQ + w\,\mathrm{d}z^2 + w^{-1}\omega^2$ is $\bigotimes_{i=1}^n (K_i^{-1} - \{0\})$.

Theorem 3.1. *Let M^{2m} be the total space of the $\mathbb{CP}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \bigotimes_{i=1}^n K_i^{-1})$ over products ${B_1}^{2m_1} \times \cdots \times {B_n}^{2m_n}$ of Kähler–Einstein manifolds B_i of scalar curvature $-2m_i$, where K_i is the canonical bundle of B_i and m_i is the complex dimension of B_i. Then M has an S^1-symmetric extremal Kähler metric.*

Remark 3.2.

1. *When $n = 1$ and $m_1 = 1$ this metric is contained in the work of Tønnesen-Friedman* [17].
2. *We could have considered a situation where each B_i has positive scalar curvature but this case is contained in the Koiso–Sakane approach* [10] *followed by Hwang* [8].

Proof of the theorem. We will show that the metric constructed above on $\bigotimes_{i=1}^n (K_i^{-1} - \{0\})$ can be compactified. We need $z > 0$, $w > 0$ and thus $P(z) > 0$. Suppose we have positive numbers $a > b$ such that $w > 0$ on the interval (a, b) and $w^{-1} = 2(z-a) + \mathrm{O}(z-a)^2$ near a; $w^{-1} = 2(b-z) + \mathrm{O}(z-b)^2$ near $z = b$. Then we can add a copy of $B_1 \times \cdots \times B_n$ at a and b to get the compact $\mathbb{CP}^1$-bundle [13,14]: for example, near $z = a$, set $r = z - a$, then $g = (1 + \mathrm{O}(r^2)(\mathrm{d}r^2 + r^2\omega^2)$. The conditions on w^{-1} can be rewritten as $w^{-1}(a) = w^{-1}(b) = 0$, $(w^{-1})'(a) = 2$, $(w^{-1})'(b) = -2$ or equivalently

$$P(a) = P(b) = 0, \quad P'(a) = 2a^{m-1}, \quad P'(b) = -2b^{m-1}. \tag{20}$$

We may rescale the metric such that $a = 1$. Thus, we need to prove the existence of $b > 1$ such that (20) is satisfied and $P(z) > 0$ on $1 < z < b$. The conditions in (20) determines the coefficients c_1, c_2, c_3, c_4 in terms of b and m. To, furthermore, secure $P(z) > 0$ on $1 < z < b$, the boundary conditions in (20) show, it is enough to prove that for each m we can find b such that $P''(z) < 0$ on $1 < z < b$, which we prove in the following lemma. $\square$

Lemma 3.3. *There exists $\beta > 1$ such that for $b \in (1, \beta)$, $P''(z)$ is negative in the interval $[1, b]$.*

Proof. We can write $P''(z) = z^{m-2} S_m(z)$ where

$$S_m(z) = -c_1(m+2)(m+1)z^2 - c_2(m+1)mz - 2(m-1).$$

One can show from (20) with $a = 1$ that if we set

$$\begin{aligned} t_1 &:= 2(-(m+1)b^{2m} + 2m^2 b^{m+1} - 2(m^2-1)b^m + (m-1)),\\ t_2 &:= 2((m+2)b^{2m+1} - 2m(m+1)b^{m+2}\\ &\quad + (2m^3 + 3m - 2)b^{m+1} - (m+2)b^m - (m-2)), \end{aligned}$$

and

$$n := m(-b^{2m+2} + (m+1)^2 b^{m+2} - 2m(m+2)b^{m+1} + (m+1)^2 b^m - 1),$$

then $c_1 = t_1/n$ and $c_2 = t_2/n$. For m fixed, c_1 and c_2 can be viewed as functions of $b > 1$. Moreover t_1, t_2 and n are clearly analytic funtions of b, also at $b = 1$.

First, observe that

$$t_1(1) = \frac{\mathrm{d}t_1}{\mathrm{d}b}(1) = \frac{\mathrm{d}^2 t_1}{\mathrm{d}b^2}(1) = 0 \quad \text{and} \quad \frac{\mathrm{d}^3 t_1}{\mathrm{d}b^3}(1) < 0.$$

This implies that there exists a constant $b_{t_1} > 1$ such that for $b \in (1, b_{t_1})$, $t_1(b) < 0$. Second, observe that

$$n(1) = \frac{\mathrm{d}n}{\mathrm{d}b}(1) = \frac{\mathrm{d}^2 n}{\mathrm{d}b^2}(1) = \frac{\mathrm{d}^3 n}{\mathrm{d}b^3}(1) = 0 \quad \text{and} \quad \frac{\mathrm{d}^4 n}{\mathrm{d}b^4}(1) < 0.$$

This implies that there exists a constant $b_n > 1$ such that for $b \in (1, b_n)$, $n < 0$. Third, consider the function $h(b) := mt_2 + (m+2)t_1$. Observe that

$$h(1) = \frac{\mathrm{d}h}{\mathrm{d}b}(1) = \frac{\mathrm{d}^2 h}{\mathrm{d}b^2}(1) = 0 \text{ and } \frac{\mathrm{d}^3 h}{\mathrm{d}b^3}(1) < 0.$$

This implies that there exists a constant $b_h > 1$ such that for $b \in (1, b_h)$, $h(b) < 0$. Now, define $\beta = \min(b_{t_1}, b_n, b_h)$. Let $b \in (1, \beta)$. Since $c_1(b) > 0$ we have that S_m is concave down. Since $h(b) < 0$ and $t_1 < 0$ the sum of roots (if any)

$$\frac{mc_2}{-(m+2)c_1} = \frac{mt_2}{-(m+2)t_1}$$

is less than 1. Since $S_m(0) = -2(m-1)$ we have that the roots of S_m, if any, will have the same sign. Therefore each root is less than 1. Thus, for $b \in (1, \beta)$, $S_m(z) < 0$ for $z \geq 1$. In particular, S_m and $P''(z)$ are negative in the interval $[1, b]$. □

Remark 3.4. *These examples are not only important because they give new extremal Kähler metrics. Also, our work gives an example of a strongly extremal Kähler metric* [9,16]. *This property is closely related to the existence of a local Einstein metric in the conformal class of* *one of the extremal Kähler metrics of Tønnesen-Friedman. The relation to Einstein geometry is a topic we shall consider in the next section.*

4. Einstein metrics in dimension four

The extremal Kähler metric of Tønnesen-Friedman in dimension four is defined on a ruled surface M (a Riemann sphere bundle over a complex curve) of genus $g > 1$ [17]. Using the fact that the homology of such a surface is generated by a fiber and a section of the projection to the Riemann surface, it is easily seen that the Euler characteristic $\chi(M)$ satisfies $\chi(M) = 4(1-g)$ (indeed, the Euler characteristic is multiplicative in fibrations). As Einstein manifolds in dimension four must have non-negative Euler characteristic [1], the Tønnesen-Friedman metric is not globally conformal to an Einstein metric. However, here we shall consider the local behavior and compare with the Calabi metric.

Consider the two triplets of one forms σ_i^ϵ given by

$$\mathrm{d}\sigma_1^\epsilon = \sigma_2^\epsilon \wedge \sigma_3^\epsilon, \quad \mathrm{d}\sigma_2^\epsilon = \sigma_3^\epsilon \wedge \sigma_1^\epsilon, \quad \mathrm{d}\sigma_3^\epsilon = \epsilon\sigma_1^\epsilon \wedge \sigma_2^\epsilon, \quad \epsilon = \pm 1. \tag{21}$$

Furthermore, consider the metrics

$$g_\epsilon = \frac{z}{4P_\epsilon(z)}(\mathrm{d}z)^2 + \frac{P_\epsilon(z)}{z}\frac{(\sigma_3^\epsilon)^2}{4} + z\frac{((\sigma_1^\epsilon)^2 + (\sigma_2^\epsilon)^2)}{4}, \tag{22}$$

where

$$P_\epsilon(z) = \epsilon z^2 - c_1 z^4 - c_2 z^3 - c_3 z - c_4. \tag{23}$$

For $\epsilon = +1$ we get Calabi's extremal metrics in the Bianchi IX class (with $U(2)$ isometry group) while for $\epsilon = -1$ we get the local version of the Tønnesen-Friedman metrics in the Bianchi VIII class (with $U(1,1)$ isometry group. In the compact metric only the circle symmetry remains after taking quotient by the discrete group).

In both cases the scalar curvature is given by

$$s = 12c_1\left(z + \frac{c_2}{2c_1}\right). \tag{24}$$

It is known [7] that if $s^3 + 6s\Delta s - 12|\mathrm{d}s|^2$ is constant, then the metric $g_{\mathrm{E}} = g_\epsilon/s^2$ is locally Einstein.

This relation gives only one constraint

$$4c_1c_4 = c_2c_3. \tag{25}$$

For fixed volume this constraint determines a unique Kähler class of g_ϵ within the range of a Tønnesen-Friedman metric. The fact that one of the extremal metrics is locally conformally Einstein implies that the extremal metric is also strongly extremal [9,16].

In order to compare with the work in the case of the Calabi metric [6], suppose that $c_2 \neq 0$, in which case we can define new coordinates and parameters by

$$r = \frac{2c_1 z - c_2}{2c_1 z + c_2}, \quad \epsilon + \frac{c_2^2}{4c_1} + \frac{c_1 c_3}{c_2} = \frac{4\lambda l^2}{3}, \quad \frac{c_2^2}{4c_1} - \frac{c_1 c_3}{c_2} = M. \tag{26}$$

Then the metric

$$g_{\mathrm{E}} = 72 l^2 c_1 c_2 \frac{g_\epsilon}{s^2} \tag{27}$$

is given by

$$g_{\mathrm{E}} = l^2 \left\{ \frac{1 - r^2}{\Delta_\epsilon(r)} (\mathrm{d}r)^2 + 4 \frac{\Delta_\epsilon(r)}{1 - r^2} (\sigma_3^\epsilon)^2 + (1 - r^2)((\sigma_1^\epsilon)^2 + (\sigma_2^\epsilon)^2) \right\}, \tag{28}$$

with

$$\Delta_\epsilon(r) = \frac{\lambda l^2}{3} r^4 + (\epsilon - 2\lambda l^2) r^2 - 2Mr + \epsilon - \lambda l^2. \tag{29}$$

The Derdzinski criteria gives that g_{E} is indeed Einstein and a direct computation shows that the Einstein constant is equal to λ. For $\epsilon = +1$ these metrics were derived by Carter [5] while for $\epsilon = -1$ we get its Bianchi VIII partner. The Calabi metric is defined on ruled surfaces of genus zero. If the degree of the line bundle is equal to 1, the manifold is the blow-up of the complex projective plane in one point and the Calabi metric is globally conformal to the Page metric [6]. The Tønnesen-Friedman metric is locally conformally Einstein but due to the vanishing of the scalar curvature (24) on the ruled surface, the conformal change does not extend to the compact surface, in agreement with the topological arguments above. However, there are complete non-compact Einstein metrics in this Bianchi VIII family (on $D \times \mathbb{R}^2$ where D is the disk in $\mathbb{R}^2$).

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